

Binary Relation

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Adapted From:

Stanford CS103 Binary Relation

《数理逻辑与集合论》9.3.4, 10.1-10.4, 10.5.1, 10.6, 10.7

Review

- **ZF Axiom System**
 - Russell's paradox in naïve set theory.
 - Subset axiom schema, Axiom of regularity
- **Cardinality of Sets**
 - Cardinality of finite sets. Principle of inclusion and exclusion.
 - Cardinality of infinite sets. Comparing cardinality.
 - Cantor's Theorem.
 - $|Problems| > |Programs|$

How we investigate computability?

2.1 Discussion on Problems

Part1. Intro & Set theory(I) : Basics & Formal Language.

Part2. Set Theory(II): Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

2.3 Discussion on computability

Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

What is the structure of a problem?

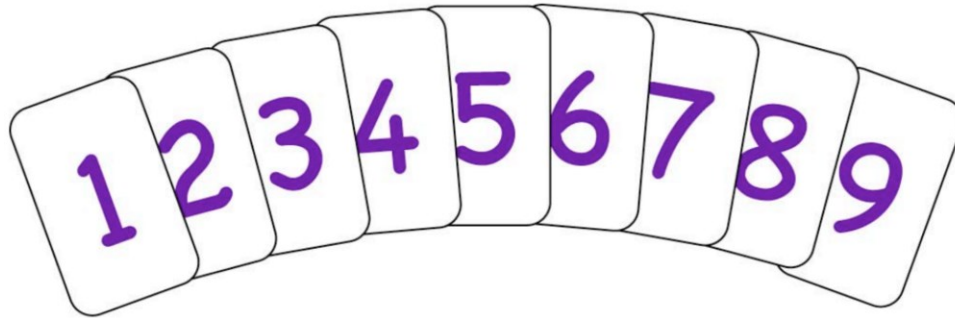
Tic Tac Toe

- Two players take turns marking the spaces in a 3×3 grid.
- The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row is the winner.

X	X	O
X	O	O
O	X	

15 Points

- Two players take turns to select a card with points from 1 to 9.
- The player who picks three cards whose sum equals 15 wins.



Take a deeper look.

X	X	O
X	O	O
O	X	

8	1	6
3	5	7
4	9	2

*They only differ in their surface structure, but
are based on an identical deep structure!!*

Structure of a Problems

- A collection of objects.
 - Cells in Tic Tac Toe.
 - Cards in 15 points.
 - Students in cdm class.
- The connection between objects.
 - Three cells are in a same horizontal, vertical, or diagonal row.
 - Sum of three cards equals 15.
 - Two students come from the same city.

Structure of a Problems

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Set Theory

Structure of a Problems

- A collection of objects.
 - Cells in Tic Tac Toe.
 - Cards in 15 points.
 - Students in cdm class.
- The connection between objects.
 - Three cells are in a same horizontal, vertical, or diagonal row.
 - Sum of three cards equals 15.
 - Two students are in same desk.

Relation & Function

What are relations?

Relations

- We've seen a lot of relations:

- Between sets:

$$A = B, A \subseteq B, A \neq B$$

- Between numbers:

$$x < y, x = y, x \equiv_k y$$

$x \equiv_k y$ means $x \% k = y \% k$

- Between cards:

$$c_1 + c_2 + c_3 = 15$$

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Connection between two objects
Binary Relation

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*Connection between three objects
Trinary Relation*

Relations

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- Between sets:

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- Between numbers:

$$x < y, x = y, x \equiv_k y$$

- Between cards:

$$c_1 + c_2 + c_3 = 15$$

- **We only focus on binary relations.**

Ordered Pair

Ordered pair $\langle a, b \rangle$ is collection of two elements a, b that has a as its first element and b as its second elements.

- Ordered pair $\langle a, b \rangle$ satisfies:
 - $a \neq b \Rightarrow \langle a, b \rangle \neq \langle b, a \rangle$
 - $\langle a, b \rangle = \langle c, d \rangle \Leftrightarrow a = c \wedge b = d$
- Examples:
 - $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$ are two different ordered pair.
 - $\langle 1, 1 \rangle$ is also an ordered pair.

Ordered Pair and Set

- We can use set to describe an ordered pair.
- Define that : $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$

- If $a = b$:

$$\langle a, b \rangle = \{\{a\}\}$$

- If $a \neq b$:

$$\langle a, b \rangle = \{\{a\}, \{a, b\}\} \neq \{\{b\}, \{b, a\}\} = \langle b, a \rangle$$

- If $\langle a, b \rangle = \langle c, d \rangle$:

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \Leftrightarrow a = c \wedge b = d$$

Cartesian Products

The Cartesian product of the sets A, B , denoted by $A \times B$, is the set of ordered pair $\langle a, b \rangle$ where $a \in A$ and $b \in B$

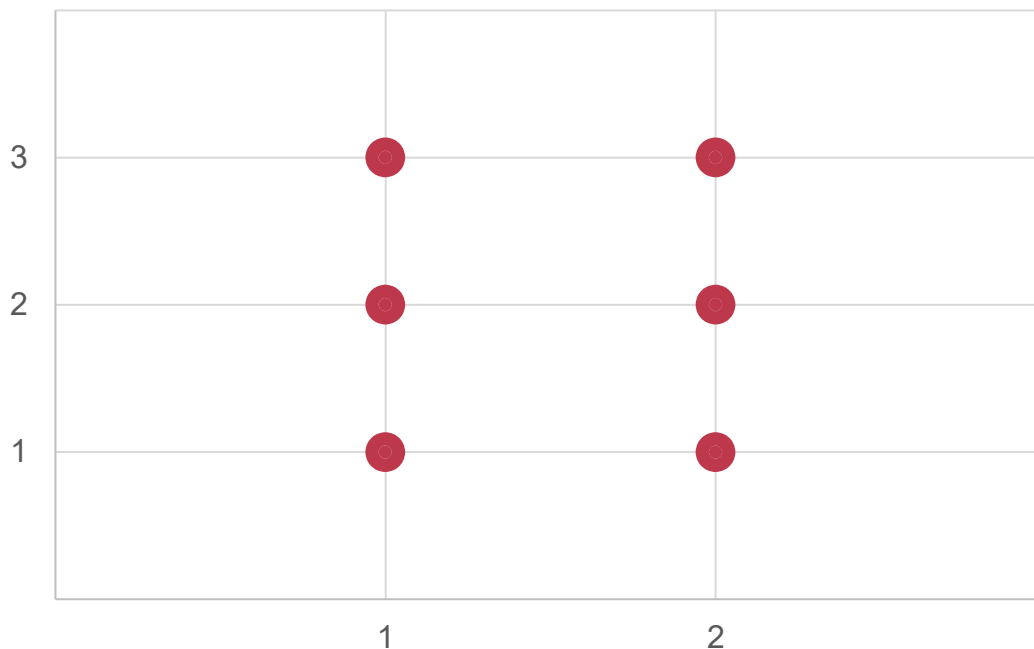
$$A \times B = \{\langle a, b \rangle | a \in A \wedge b \in B\}$$

- Let $A = \{1,2\}, B = \{1,2,3\}$
 - $\emptyset \times A = B \times \emptyset = \emptyset$
 - $A \times B = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle\}$
 - $B \times A = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle, \langle 3,2 \rangle\} \neq A \times B$

Cartesian Products

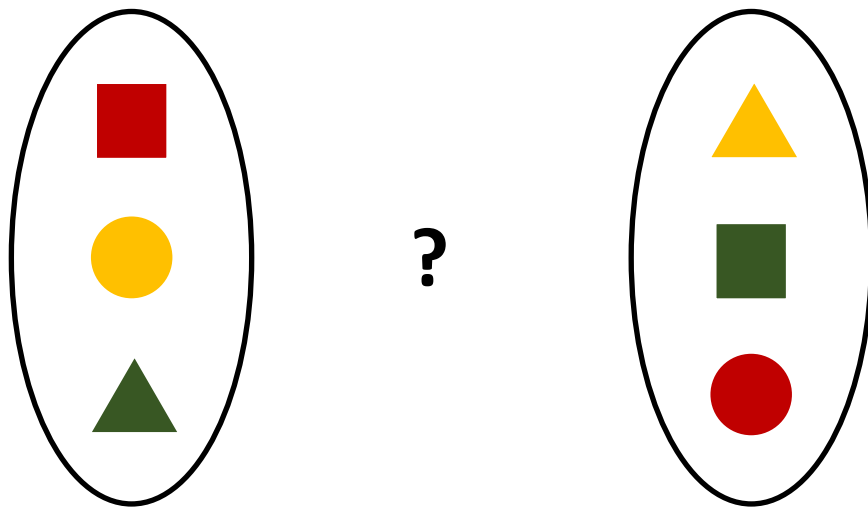
$$A \times B$$

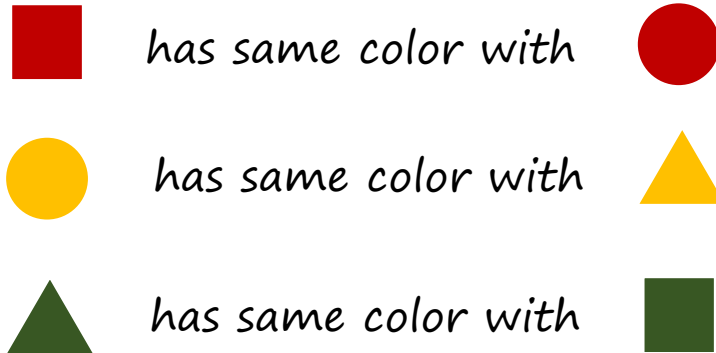
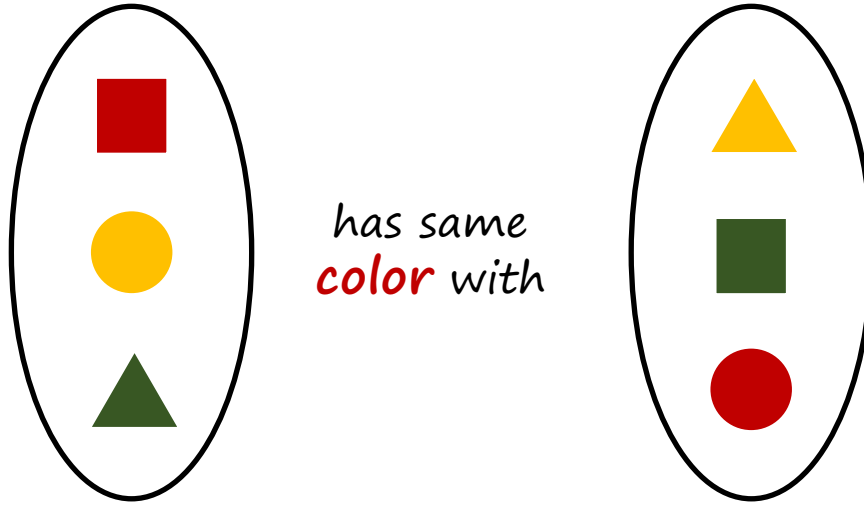
$$B = \{1,2,3\}$$

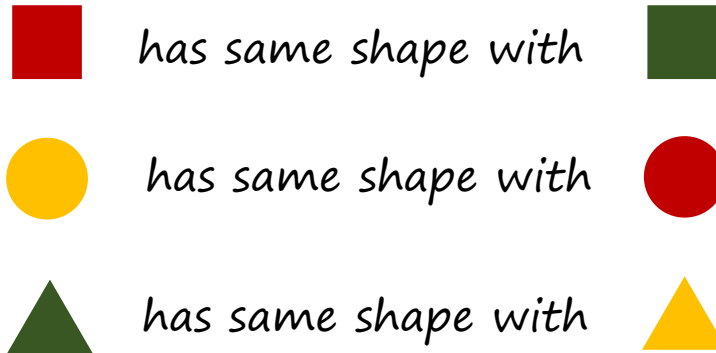
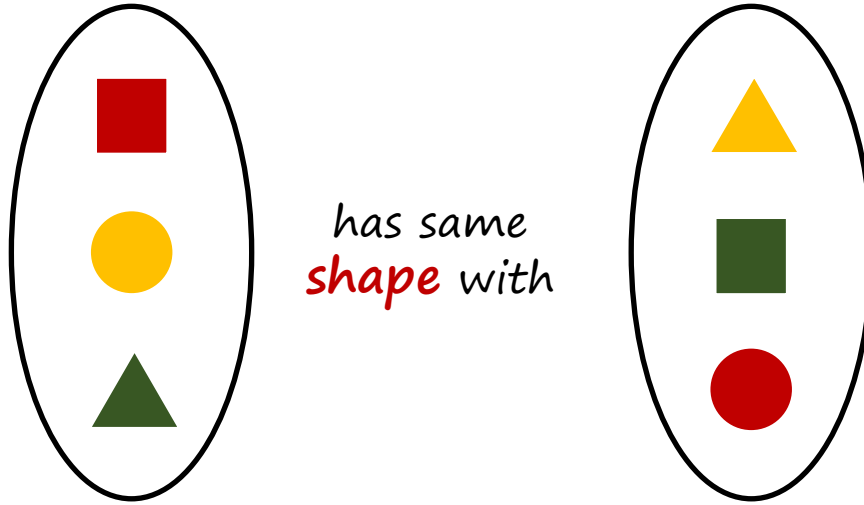


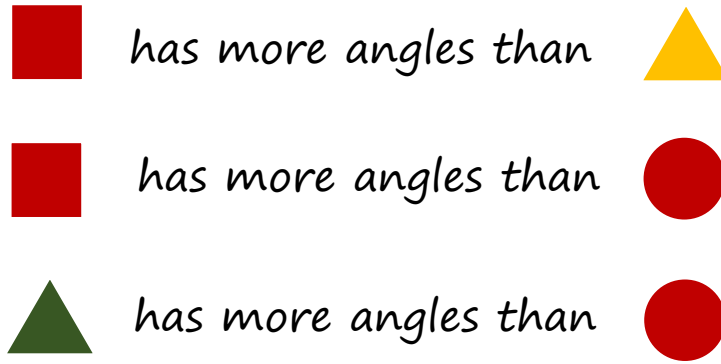
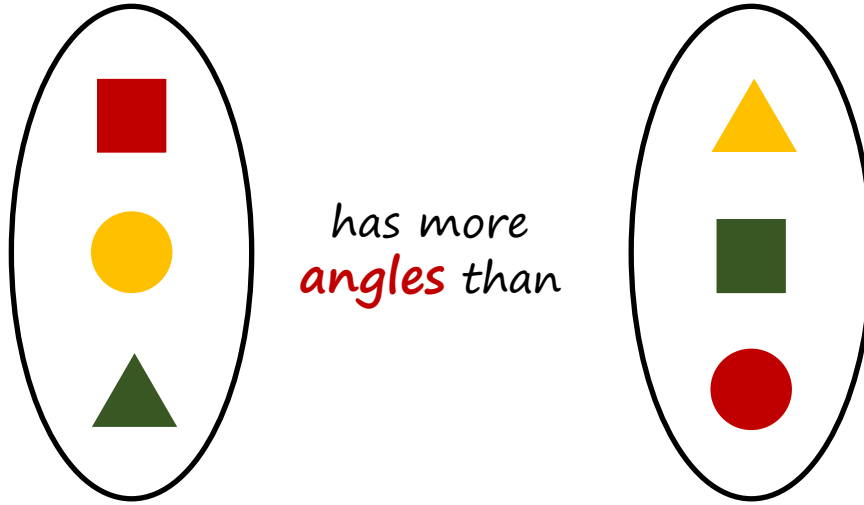
$$A = \{1,2\}$$

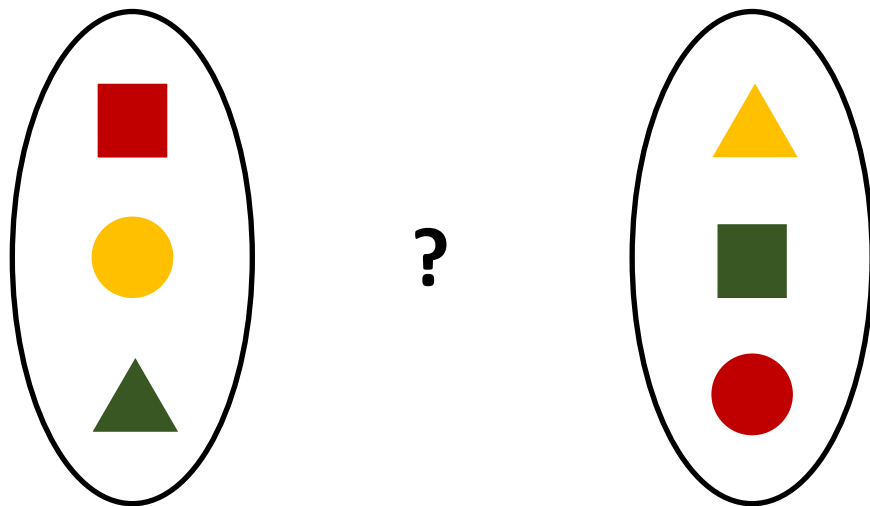
What exactly is a binary relation?











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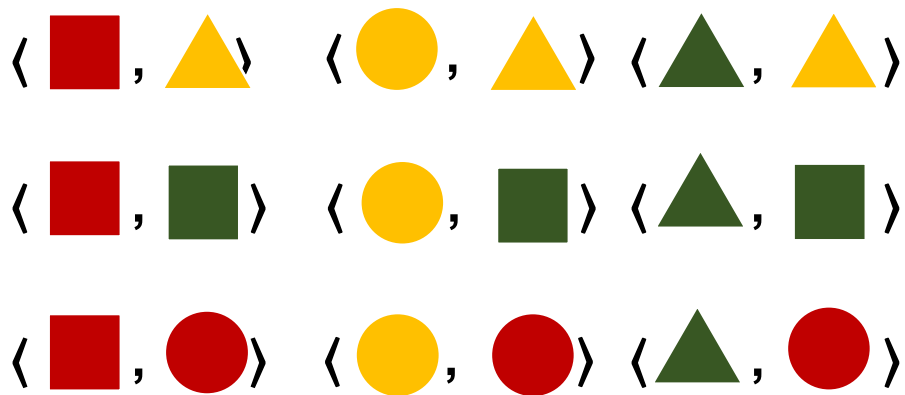
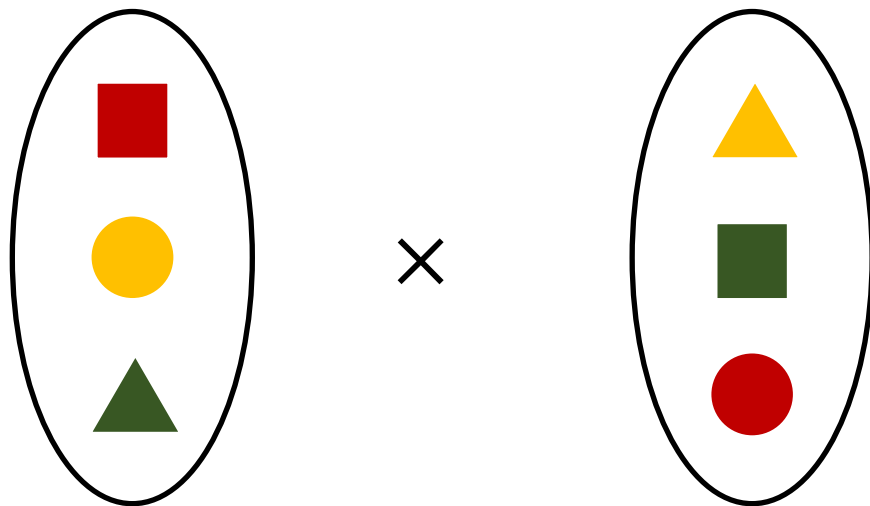


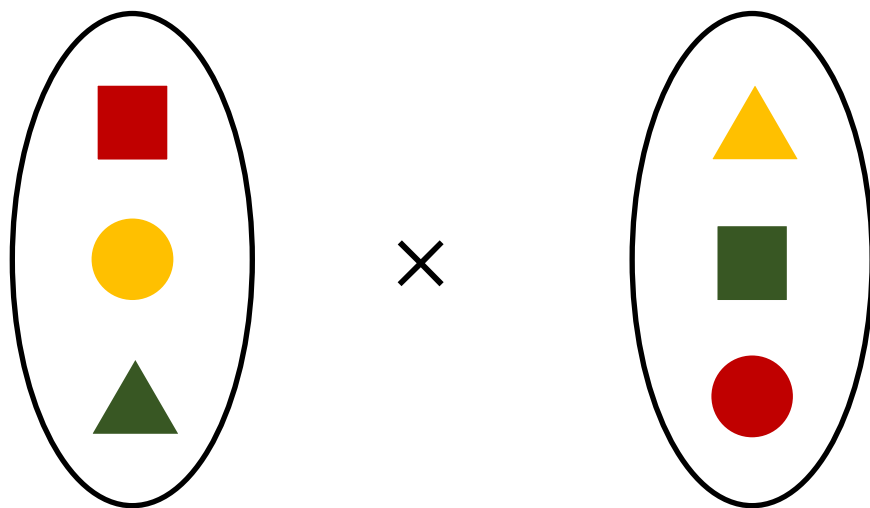
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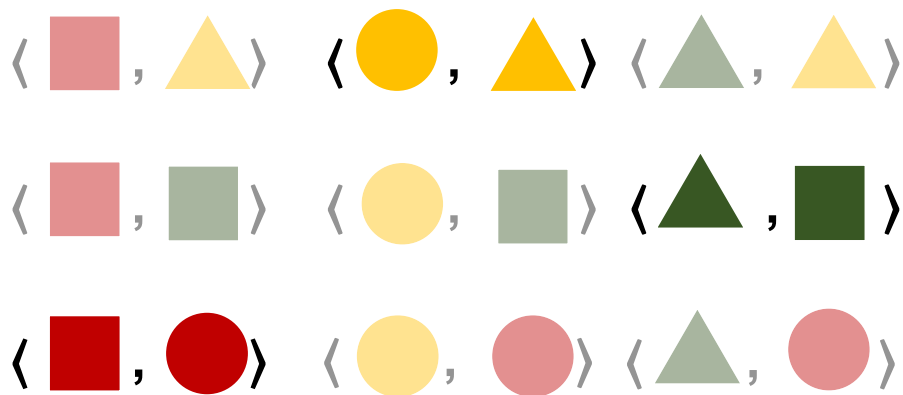
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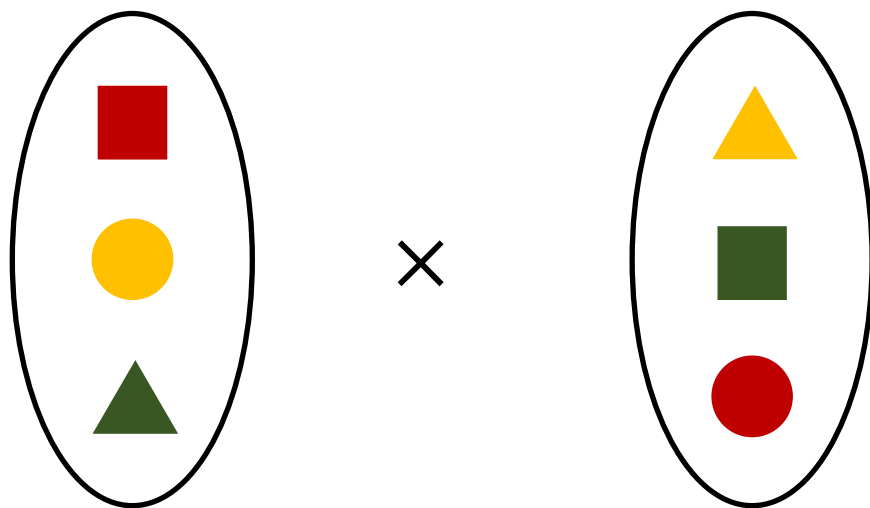




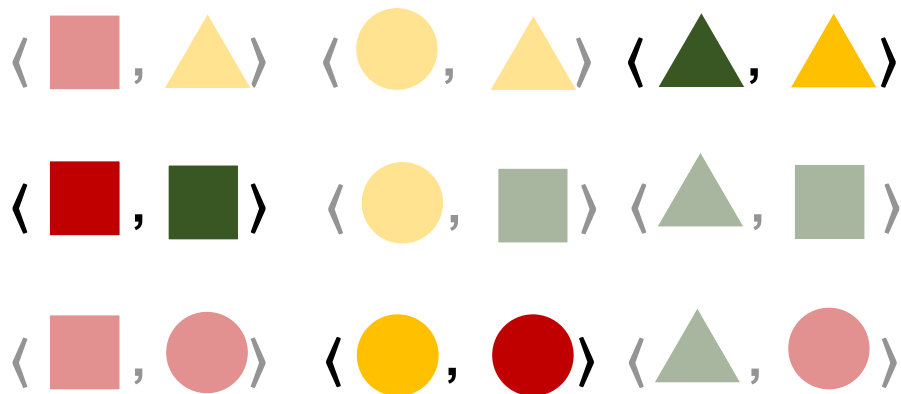


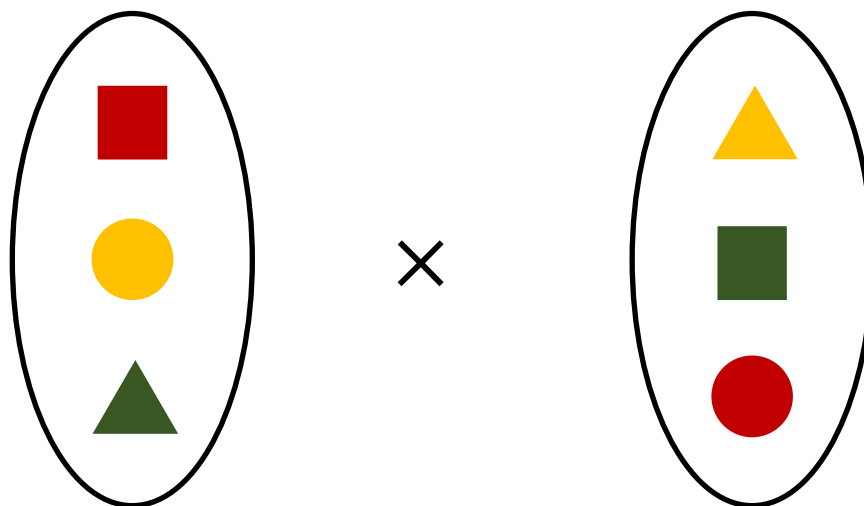
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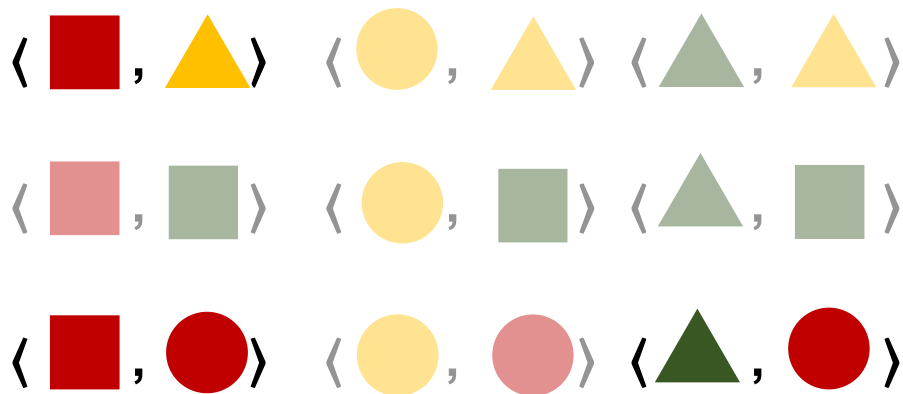


has same *shape* with





has more *angles* than



Binary Relation

Let A, B be sets. A binary relation from A to B is a subset of $A \times B$.

- In other words, a binary relation from A to B is a set R of **ordered pairs**, where the first element of each ordered pair comes from A and the second element comes from B .
 - aRb means $\langle a, b \rangle \in R$
 - $a \not R b$ means $\langle a, b \rangle \notin R$
- A relation on a set A is a relation from A to A , or is a subset of $A \times A$.

Domain, Range and Field

- For a relation ***R*** from ***A*** to ***B***, we define:

- The domain(定义域)of *R* is:

$$dom(R) = \{x | (\exists y)(\langle x, y \rangle \in R)\}$$

- The range(值域)of *R* is:

$$ran(R) = \{y | (\exists x)(\langle x, y \rangle \in R)\}$$

- The field(域)of *R* is:

$$fld(R) = dom(R) \cup ran(R)$$

Binary Relation

- For set $A = \{1,2,3\}$, define:
 - $R_1 = \{\langle x, y \rangle | x \in A \wedge y \in A \wedge x|y\}$
 - $R_2 = \{\langle x, y \rangle | x \in A \wedge y \in A \wedge x < y\}$
- Then
 - $1R_12, 2R_23$
 - $R_1 = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle\}$
 - $R_2 = \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,3 \rangle\}$
 - $dom(R_1) = \{1,2,3\}, ran(R_1) = \{1,2,3\}, fld(R_1) = \{1,2,3\}$
 - $dom(R_2) = \{1,2\}, ran(R_2) = \{2,3\}, fld(R_2) = \{1,2,3\}$

Special Relations

- For any set A , there are three kinds of special relations.

- The identity relation (恒等关系) :

$$I_A = \{\langle x, x \rangle | x \in A\}$$

- The universal relation (全域关系) :

$$E_A = \{\langle x, y \rangle | x \in A \wedge y \in A\}$$

- The empty relation (空关系) :

$$N_A = \emptyset$$

- For arbitrary elements $x \in A$ and $y \in A$:

- $xN_A y$ holds never
- $xE_A y$ holds always
- $xI_A y$ holds if and only if $x = y$

Relation Matrix(关系矩阵)

- Suppose that R is a relation from $A = \{a_1, a_2, a_3, \dots, a_m\}$ to $B = \{b_1, b_2, b_3, \dots, b_n\}$. Then the relation R can be represented by zero-one matrix $M_R = (m_{ij})_{m \times n}$, where:

$$m_{ij} = \begin{cases} 1 & \text{if } \langle a_i, b_j \rangle \in R \\ 0 & \text{if } \langle a_i, b_j \rangle \notin R \end{cases}$$

- If R is a relation on $A = \{a_1, a_2, a_3, \dots, a_m\}$, then the relation matrix of R is a $m \times m$ square $M_R = (m_{ij})_{m \times m}$

Relation Matrix: Example

Let $A = \{1, 2, 3, 4\}$, the relation R_1 on A is:

$$R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle\}$$

The relation matrix of R_1 is:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix: Example

Identity relation I :

$$M_I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Relation Matrix: Example

Universal relation E :

$$M_E = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

Relation Matrix: Example

Empty relation N :

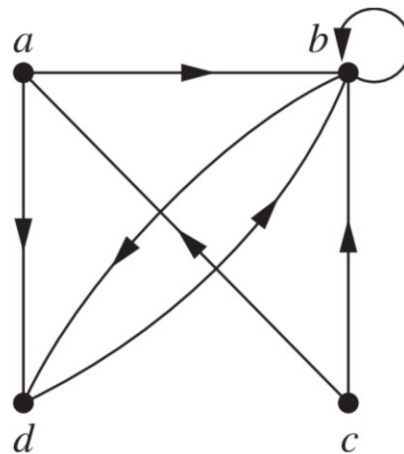
$$M_N = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Digraphs(关系图)

Defination

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b) , and the vertex b is called the terminal vertex of this edge.

The directed graph with vertices a, b, c , and d , and edges $\langle a, b \rangle, \langle a, d \rangle, \langle b, b \rangle, \langle b, d \rangle, \langle c, a \rangle, \langle c, b \rangle$, and $\langle d, b \rangle$



Digraphs

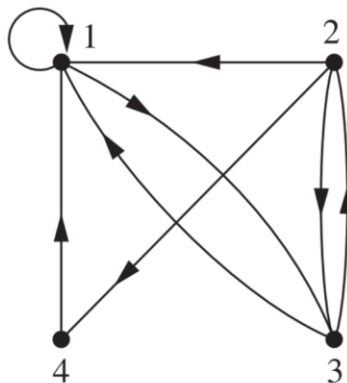
Digraph of Relation R on set A :

- Vertices : Elements of A
- Edges: Ordered pairs in R

The directed graph of the relation

$$R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle\}$$

on the set $\{1, 2, 3, 4\}$ is:



Combining Relations

- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Let $A = \{1, 2, 3\}$. The relations $R_1 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 3, 2 \rangle\}$ and $R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\}$ can be combined to obtain:

- $R_1 \cup R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$
- $R_1 \cap R_2 = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle\}$
- $R_1 - R_2 = \{\langle 2, 1 \rangle, \langle 3, 2 \rangle\}$
- $R_2 - R_1 = \{\langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\}$
- $R_1 \oplus R_2 = \{\langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$

Combining Relations: Relation Matrix

Or represent in matrix:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{then:}$$

$$\begin{aligned} M_{R_1 \cup R_2} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, & M_{R_1 \cap R_2} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M_{R_1 - R_2} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & M_{R_2 - R_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ M_{R_2 \oplus R_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

Inverse of Relations (关系的逆)

Defination

Let R be a relation from a set A to a set B . The inverse relation of R , denoted by R^{-1} , is the set of ordered pairs $\{\langle b, a \rangle \mid \langle a, b \rangle \in R\}$ from B to A

Let $A = \{a, b, c\}$, a relation on A , $R = \{\langle a, b \rangle, \langle a, c \rangle, \langle b, a \rangle, \langle c, b \rangle\}$, then:

$$R^{-1} = \{\langle b, a \rangle, \langle c, a \rangle, \langle a, b \rangle, \langle b, c \rangle\}$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Inverse of Relations: Relation Matrix

R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then :

$$M_{R^{-1}} = M_R^T, \text{ where } M_R^T \text{ is the transpose of } M_R$$

Proof:

Assume that $M_R = \langle s_{ij} \rangle_{m \times n}$ where s_{ij} is the element. And $M_{R^{-1}} = \langle t_{ij} \rangle_{n \times m}$, where t_{ij} is the element.

If $s_{ij} = 1$, then we know $\langle a_i, b_j \rangle \in R$, according to the definition of inverse, we have $\langle b_j, a_i \rangle \in R^{-1}$, so $t_{ji} = 1$

If $s_{ij} = 0$, then we know $\langle a_i, b_j \rangle \notin R$. So $\langle b_j, a_i \rangle \notin R^{-1}$, otherwise $\langle a_i, b_j \rangle \in R$, then $t_{ji} = 0$

In any case, we have $s_{ij} = t_{ji}$, and it means M_R and $M_{R^{-1}}$ are transposed each other. Q.E.D

Composite of Relations (关系的合成)

Definition

Let R be a relation from A to B , and S be a relation from B to C . The composite of R and S , denote as $S \circ R$, is:

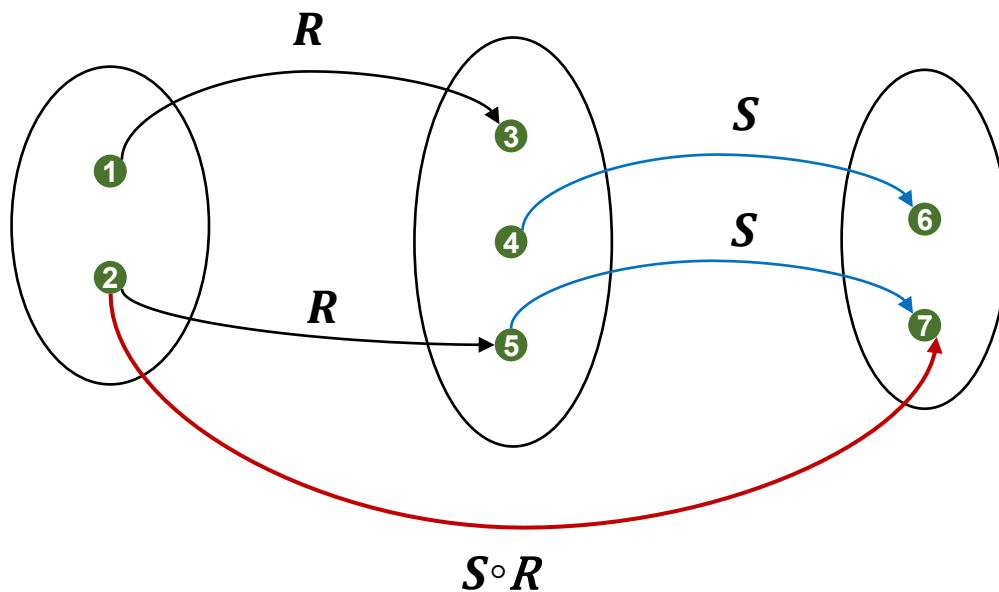
$$S \circ R = \{\langle x, y \rangle | (\exists z)(\langle x, z \rangle \in R \wedge \langle z, y \rangle \in S)\}$$

Let $A = \{1, 2, 3\}$, two relations on A ,

$R = \{\langle 1, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$, $S = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 3 \rangle\}$

Then $S \circ R = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle\}$

Composite of Relations



Composite of Relations: Relation Matrix

Let A be a set and R and S are two relations on A , the relation matrix of R and S are:

$$M_R = (r_{ij}), M_S = (s_{ij})$$

Then we have:

$$M_{S \circ R} = M_R \odot M_S = (w_{ij})$$

where $\odot$ is the matrix boolean product and $w_{ij} = \bigvee_{k=1}^n (r_{ik} \wedge s_{kj})$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{1} \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} \boxed{1} & 1 \\ \boxed{0} & 0 \\ \boxed{0} & 1 \end{bmatrix} = \begin{bmatrix} \boxed{(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{1} \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & \boxed{1} \\ 0 & \boxed{0} \\ 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} & 1 & \boxed{1} \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ \textcolor{brown}{0} & \textcolor{red}{1} & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & \textcolor{brown}{1} \\ 0 & \textcolor{red}{0} \\ 0 & \textcolor{blue}{1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Property of Inverse and Composite

① $\text{dom}(R^{-1}) = \text{ran}(R)$

② $\text{ran}(R^{-1}) = \text{dom}(R)$

③ $(R^{-1})^{-1} = R$

④ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

⑤ $(R \circ S) \circ Q = R \circ (S \circ Q)$

⑥ $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3$

⑦ $(R_1 \cup R_2) \circ R_3 = R_1 \circ R_3 \cup R_2 \circ R_3$

⑧ $R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$

⑨ $(R_1 \cap R_2) \circ R_3 \subseteq R_1 \circ R_3 \cap R_2 \circ R_3$


$$R_1 \circ (R_2 \cap R_3) \subseteq R_1 \circ R_2 \cap R_1 \circ R_3$$

Proof:

$$\langle x, y \rangle \in R_1 \circ (R_2 \cap R_3)$$

$$\Leftrightarrow (\exists z)(\langle x, z \rangle \in (R_2 \cap R_3) \wedge \langle z, y \rangle \in R_1)$$

$$\Leftrightarrow (\exists z)(\langle x, z \rangle \in R_2 \wedge \langle x, z \rangle \in R_3 \wedge \langle z, y \rangle \in R_1)$$

$$\Rightarrow (\exists z)(\langle x, z \rangle \in R_2 \wedge \langle z, y \rangle \in R_1) \wedge (\exists z)(\langle x, z \rangle \in R_3 \wedge \langle z, y \rangle \in R_1)$$

$$\Leftrightarrow \langle x, y \rangle \in (R_1 \circ R_2) \wedge \langle x, y \rangle \in (R_1 \circ R_3)$$

$$\Leftrightarrow \langle x, y \rangle \in (R_1 \circ R_2) \cap (R_1 \circ R_3)$$

Power of Relations (关系的幂)

- For a relation R on set A , $n \in \mathbb{N}$, the n th power of R , denoted by R^n , is :
 - $R^0 = I_A$
 - $R^{n+1} = R^n \circ R$
- $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^0 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle\}$
 - $R^1 = R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^2 = R^4 = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle\}$
 - $R^3 = R^5 = \{\langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle\}$

Power of Relations

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 - $R^0 = I_A$
 - $R^{n+1} = R^n \circ R$
- $R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^0 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle\}$
 - $R^1 = R = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle\}$
 - $R^2 = R^4 = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle\}$
 - $R^3 = R^5 = \{\langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle\}$

Some different power of R are equal.

Power of Relations

A is a finite set and $|A| = n$, R is a relation on A , then there exists natural number $s, t, s \neq t$, such that $R^s = R^t$

Proof:

Any relation on A is a subset of $A \times A$, or an element of $\mathcal{P}(A \times A)$.

We know that $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{|A|^2} = 2^{n^2}$, so there are finite many relations on a finite set A . But there are infinite many powers of R . It means there are different powers of R which are equal, or

$$(\exists s)(\exists t)(s \in \mathbb{N} \wedge t \in \mathbb{N} \wedge s \neq t \wedge R^s = R^t)$$

Q.E.D

Power of Relations

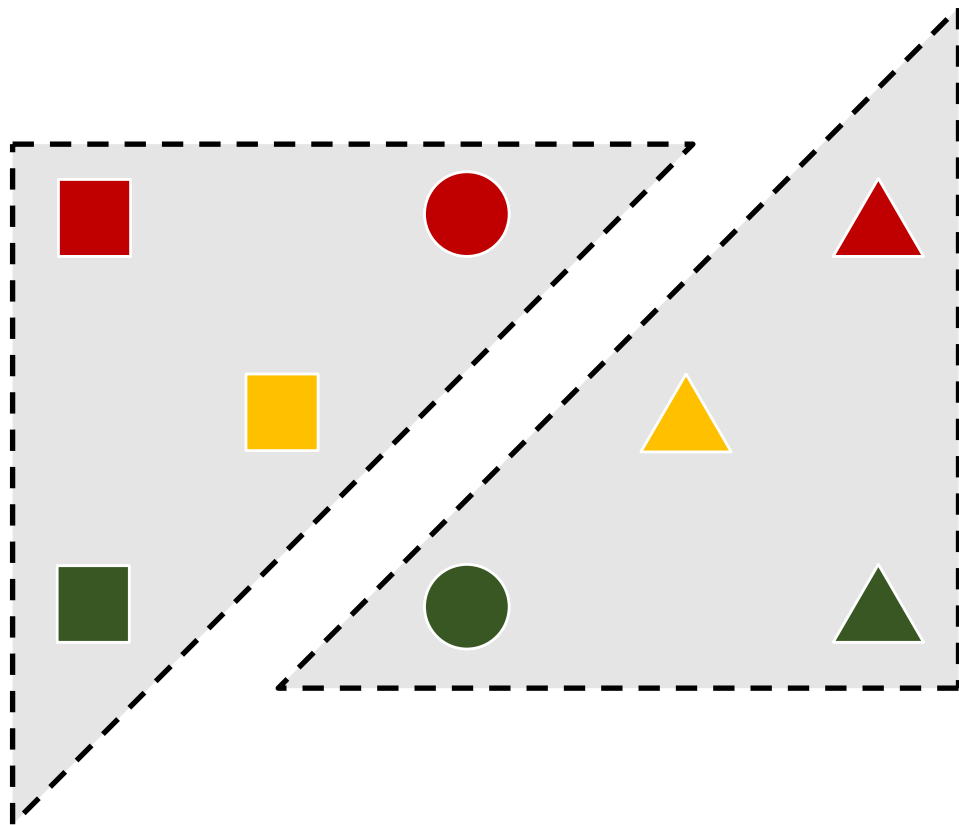
- $R^m \circ R^n = R^{m+n}$
- $(R^m)^n = R^{mn}$
- A is a finite set and $|A| = n$, R is a relation on A , and there exists natural number $s, t, s \neq t$, such that $R^s = R^t$
 - $R^{s+k} = R^{t+k}, k \in \mathbb{N}$
 - $R^{s+kp+i} = R^{t+kp+i}, k \in \mathbb{N}, i \in \mathbb{N}, p = t - s$
 - Let $B = \{R^0, R^1, \dots, R^{t-1}\}$, then for any natural number $q, R^q \in B$

How can we capture a structure with relations?

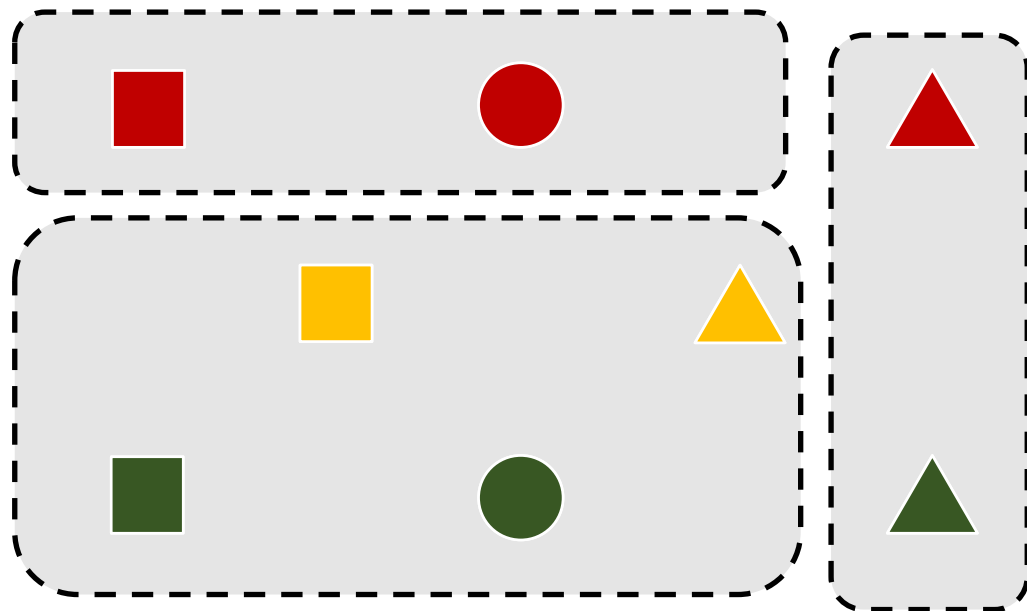
Partitions(划分)



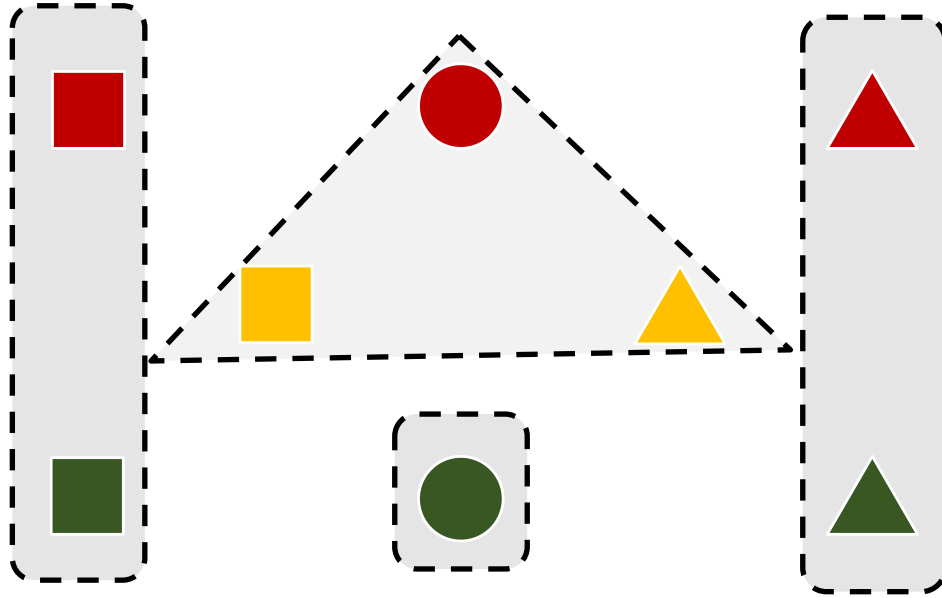
Partitions



Partitions



Partitions



Partitions

- A **partition of a set** is a way of splitting the set into **disjoint, nonempty subsets** so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.

Partitions

- More formally, for a non-empty set A , a set π is called the partition of A if:
 - $\emptyset \notin \pi$
 - $(\forall x)(x \in \pi \rightarrow x \subseteq A)$
 - $\bigcup \pi = A$
 - $(\forall x)(\forall y)((x \in \pi \wedge y \in \pi \wedge x \neq y) \rightarrow x \cap y = \emptyset)$

Partitions

- For set $A = \{a, b, c, d\}$
 - $\pi_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $\pi_2 = \{\{a, b, c, d\}\}$
 - $\pi_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $\pi_4 = \{\{a, b, d\}\}$
- } valid partitions*
- } not partition*

What is the relationship between partitions and relations?

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle \mid (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:
 - aRa for $\forall a \in A$
 - If aRb , then bRa
 - If aRb and bRc , then aRc

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle | (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:

- aRa for $\forall a \in A$

An element is in the same partition block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- If aRb and bRc , then aRc

If a and b are in the same block, b and c are also in the same block, then a and c are in the same block which contains b .

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

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Reflexivity(自反性)

- Some relations always hold from any element to itself.
- Examples:
 - $x = x$ for any x
 - $A \subseteq A$ for any set A
 - $x \geq x$ for any x
- Relations of this sort are called reflexive.

R on A is reflexive $\Leftrightarrow (\forall a)(a \in A \rightarrow aRa)$

Reflexivity: Example

- I_A and E_A
- $=, \geq, \leq$ on $\mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$

Reflexivity: Example

- I_A and E_A 😊
 - $aI_A a$ and $aE_A a$ for all $a \in A$
- $=, \geq, \leq$ on $\mathbb{R}$ 😊
 - $a = a, a \geq a, a \leq a$ for all $a \in \mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$ 😞
 - $\langle 3,3 \rangle \notin R$, it's not reflexive

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Symmetry(对称性)

- In some relations, the relative order of the objects doesn't matter.
- Examples:
 - If $x = y$ then $y = x$
 - If $x \equiv_k y$ then $y \equiv_k x$
- These relations are called symmetric.

$$\begin{aligned} & \textbf{R on A is symmetric} \Leftrightarrow \\ & (\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa) \end{aligned}$$

Symmetry : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Symmetry : Example

- I_A and E_A 😊
 - $aI_Ab \rightarrow bI_Aa$, $aE_Ab \rightarrow bE_Aa$
- N_A 😊
 - $aN_Ab \rightarrow bN_Aa$ always hold cause aN_Ab never holds
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😊

Symmetry : Relation Matrix

R is symmetric $\Leftrightarrow M_R$ is symmetric matrix

- So if R is symmetric , then $R^{-1} = R$ as $M_{R^{-1}} = M_R^T = M_R$
- M_{E_A} , M_{I_A} and M_{N_A} are both symmetric matrix

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity(传递性)

- Many relations can be chained together.
- Examples:
 - If $x = y$ and $y = z$ then $x = z$
 - If $P \subseteq Q$ and $Q \subseteq S$, then $P \subseteq S$
 - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$
- These relations are called transitive.

R on A is transitive $\Leftrightarrow$

$$(\forall a)(\forall b)(\forall c)((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Transitivity : Example

- I_A and E_A 😊
- N_A 😊
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😞
 - $\langle 1,2 \rangle \in R, \langle 2,3 \rangle \in R$ but $\langle 1,3 \rangle \notin R$

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are reflexive, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also reflexive
- If R_1, R_2 are symmetric, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also symmetric
- If R_1, R_2 are transitive, then $R_1^{-1}, R_1 \cap R_2$ are also transitive

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are symmetric, then $R_1 \cup R_2$ are also symmetric.

Proof:

$$\langle x, y \rangle \in R_1 \cup R_2$$

$$\Leftrightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2$$

$$\Leftrightarrow \langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2$$

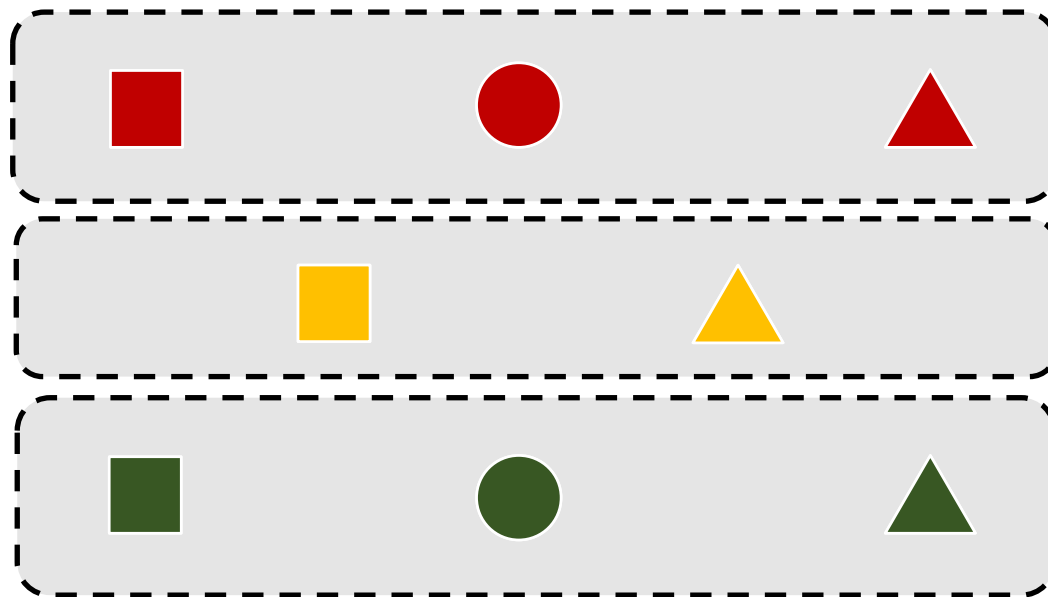
$$\Leftrightarrow \langle y, x \rangle \in R_1 \cup R_2$$

Q.E.D

Equivalence Relations(等价关系)

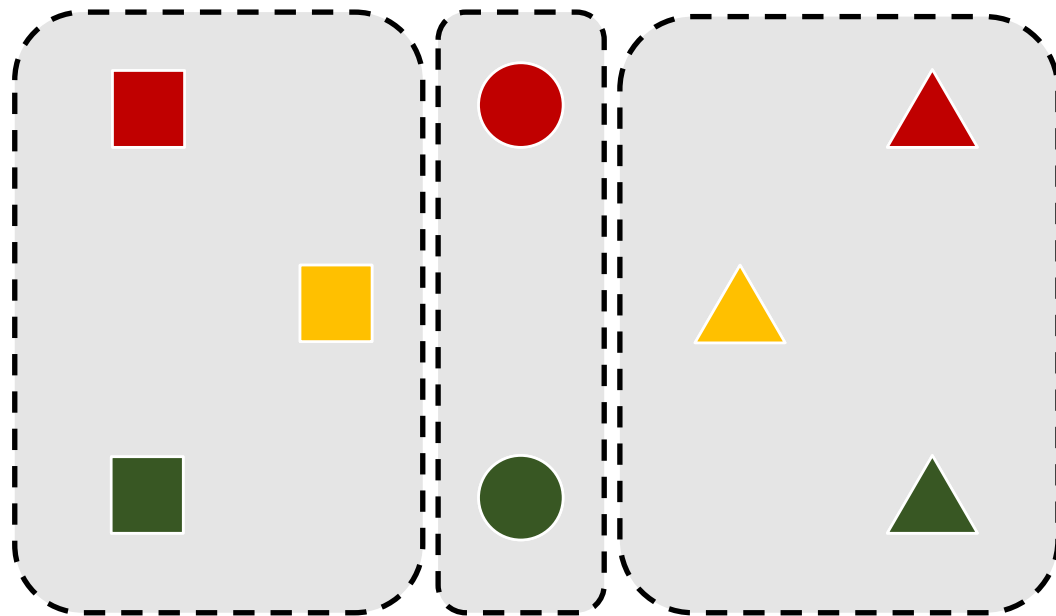
- An **equivalence relation** is a relation that is **reflexive, symmetric and transitive**.
- Some examples:
 - $x = y$
 - $x \equiv_k y$
 - x has the same color with y
 - x has the same shape with y

Partitions



aRb if a has the *color* with b

Partitions



aRb if a has the *shape* with b

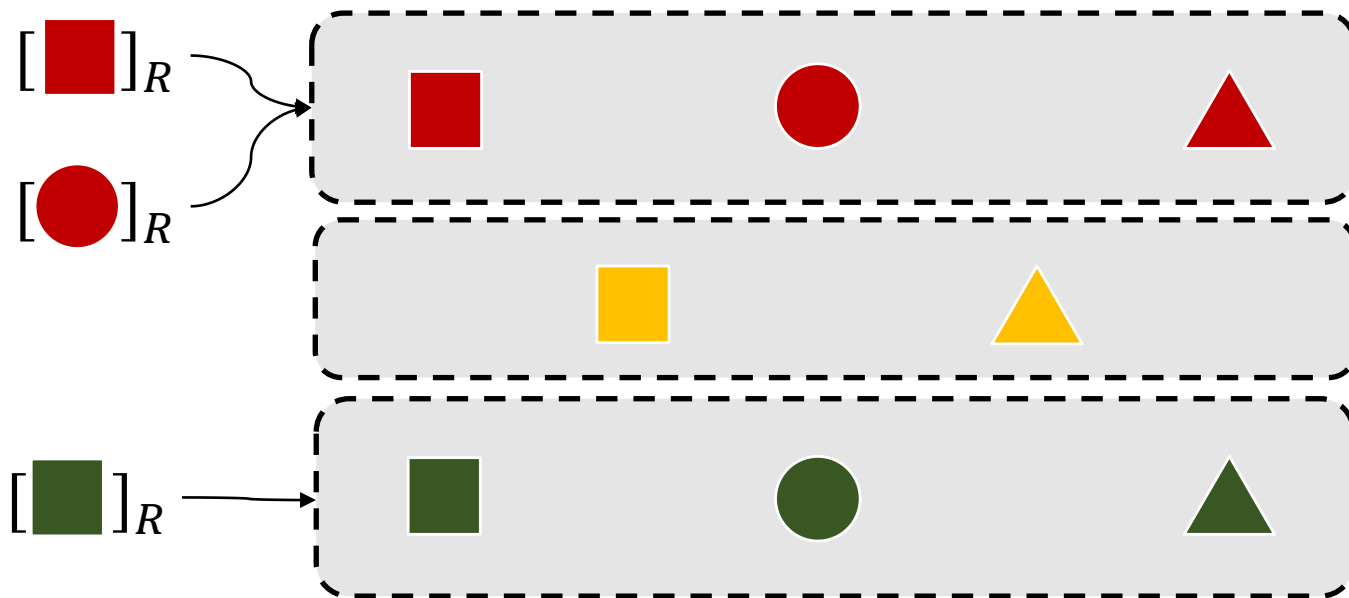
Equivalence Classes(等价类)

- Given an equivalence relation R over a set A , for any $x \in A$, the **equivalence class of x** is the set:

$$[x]_R = \{y | y \in A \wedge xRy\}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R .

Partitions



aRb if a has the *color* with b

Quotient Set(商集)

- Given an equivalence relation R over a set A , the quotient set of A is set of all the disjoint equivalence classes of A , denoted by A/R :

$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

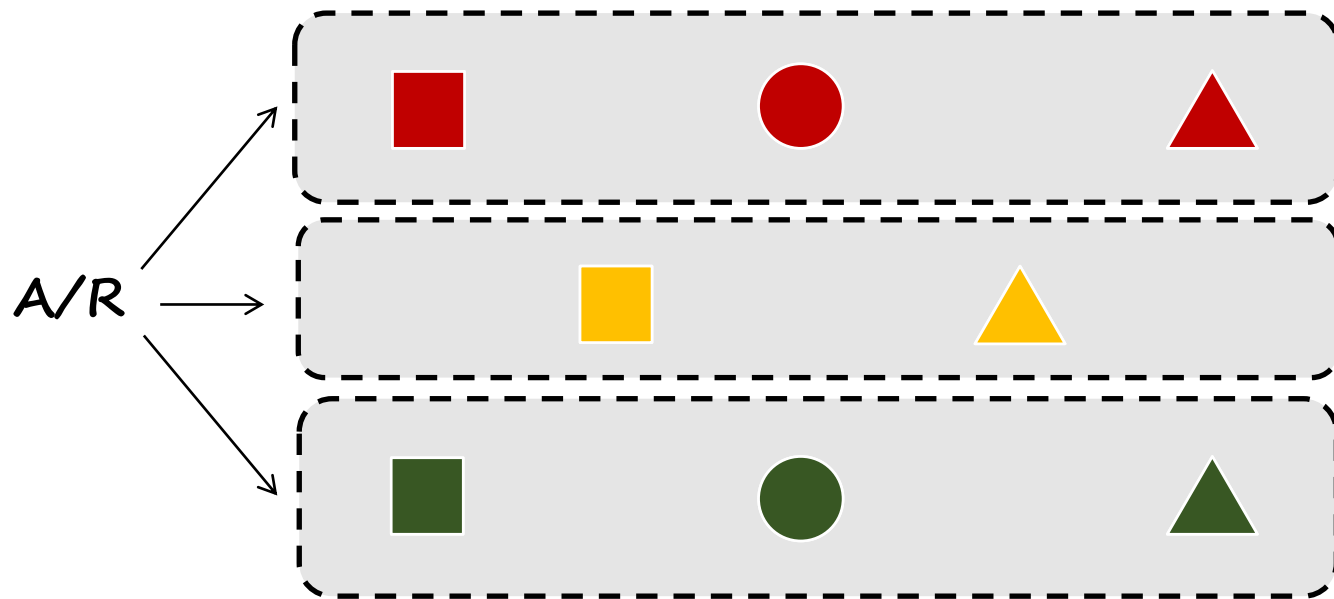
Quotient Set(商集)

- Given an equivalence relation R over a set A , the quotient set of A is set of all the disjoint equivalence classes of A , denoted by A/R :

$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

- For $R = \{\langle x, y \rangle | x \equiv_3 y\}$ on set $A = \{1, 2, 3 \dots 8\}$
 - $A/R = \{[1]_R, [2]_R, [3]_R\} = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6\}\}$

Partitions



aRb if a has the **color** with b

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$$\emptyset \notin A/R$$

$$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$$

$$\cup A/R = A$$

$$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ because $x \in [x]_R$ —→

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

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Quotient Set and Partition

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Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ cause $x \in [x]_R$

$[x]_R \subseteq A$

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

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For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ cause $x \in [x]_R$

$[x]_R \subseteq A$

$x \in [x]_R$

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$\bigcup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$[x]_R \neq \emptyset$ cause $x \in [x]_R \longrightarrow \emptyset \notin A/R$

$[x]_R \subseteq A \longrightarrow (\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$x \in [x]_R \longrightarrow \cup A/R = A$

$[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset \longrightarrow \langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Equivalence Relation and Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition.

$$R = \{\langle a, b \rangle \mid \pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0\}$$

A partition can generate an equivalence relation.

- Given an equivalence relation R over a set A , then A/R is a partition of A

An equivalence relation can generate a partition.

*Equivalence relations precisely capture what it means
to be a partition.*

Binary relations give us a *common language* to describe *common structures*.

Equivalence Relation In Real Life

- Most modern programming languages include some sort of hash table data structure.
 - **Java: HashMap**
 - **C++: std::unordered_map**
 - **Python: dict**
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relation In Real Life

- “The equals method implements an equivalence relation on non-null object references:
 - It is reflexive: for any non-null reference value x , $x.equals(x)$ should return true.
 - It is symmetric: for any non-null reference values x and y , $x.equals(y)$ should return true if and only if $y.equals(x)$ returns true.
 - It is transitive: for any non-null reference values x , y , and z , if $x.equals(y)$ returns true and $y.equals(z)$ returns true, then $x.equals(z)$ should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

- “The equals method implements **an equivalence relation on non-null object references**:
 - **It is reflexive**: for any non-null reference value *x*, *x.equals(x)* should return true.
 - **It is symmetric**: for any non-null reference values *x* and *y*, *x.equals(y)* should return true if and only if *y.equals(x)* returns true.
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Java 8 Documentation

Equivalence Relation In Real Life

“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements $\langle 17.6.3.4 \rangle$ and acts as a hash function for argument values of type *Key*, and by a binary predicate *Pred* that induces an equivalence relation on values of type *Key*. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type *T* with the *Key*.”

C++14 ISO Spec, §23.2.5/3

Equivalence Relation In Real Life

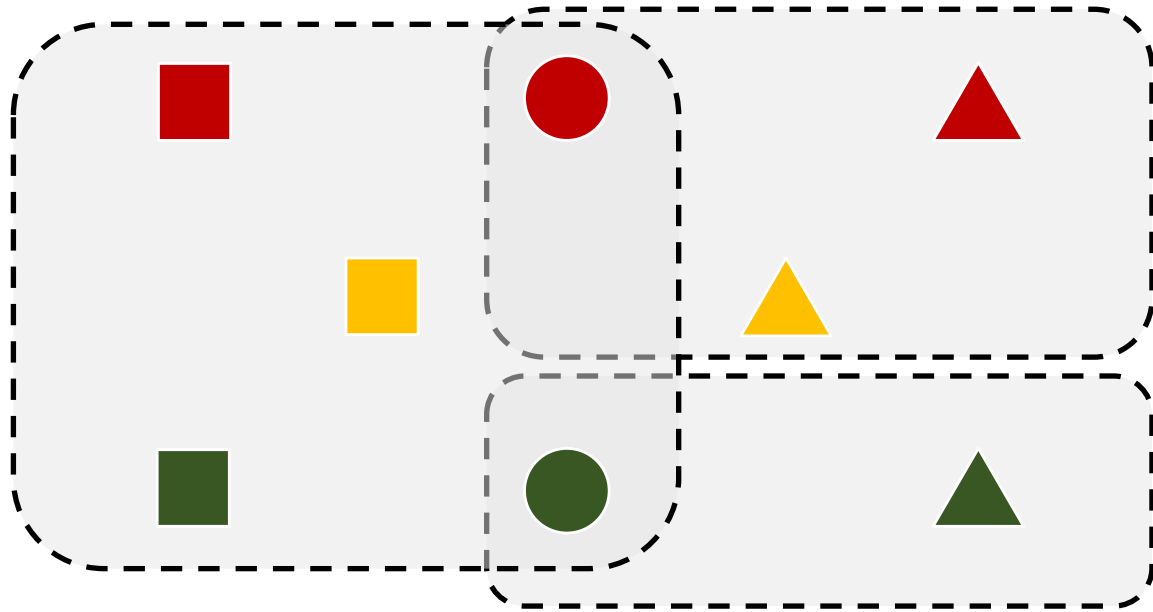
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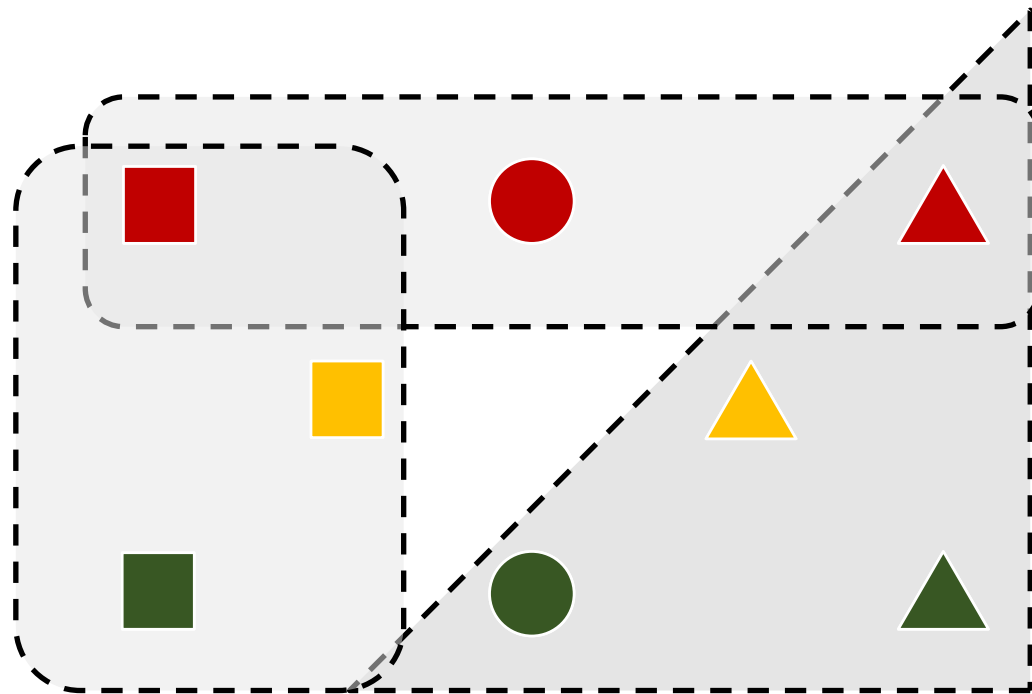
Covers (覆盖)



Covers



Covers



Covers

- A **cover of a set** is a way of splitting the set into **nonempty subsets** so that every element belongs to at least one subset.
- Intuitively, a cover contains pieces that cover every element of a set.
- **A partition is a cover, but a cover is not a partition.**

Covers

- More formally, for a non-empty set A , a set $\mathcal{C}$ is called the cover of A if:
 - $\emptyset \notin \mathcal{C}$
 - $(\forall x)(x \in \mathcal{C} \rightarrow x \subseteq A)$
 - $\bigcup \mathcal{C} = A$
 - ~~– $(\forall x)(\forall y)((x \in \mathcal{C} \wedge y \in \mathcal{C} \wedge x \neq y) \rightarrow x \cap y = \emptyset)$~~

Covers

- For set $A = \{a, b, c, d\}$
 - $C_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $C_2 = \{\{a, b, c, d\}\}$
 - $C_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $C_4 = \{\{a, b, d\}\}$
- } valid covers
- } not a cover

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists \mathcal{C}_0)(\mathcal{C}_0 \in \mathcal{C} \wedge a \in \mathcal{C}_0 \wedge b \in \mathcal{C}_0)\}$$

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists \mathcal{C}_0)(\mathcal{C}_0 \in \mathcal{C} \wedge a \in \mathcal{C}_0 \wedge b \in \mathcal{C}_0)\}$$

- Then we have:
 - aRa for all $a \in A$
 - If aRb , then bRa
 - ~~If aRb and bRc , then aRc~~

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:

- aRa for all $a \in A$

An element is in the same block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- ~~– If aRb and bRc , then aRc~~

If a and b are in the same block C_0 , b and c are also in the same block $C_1 \neq C_0$, a and b are not in the same block.

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

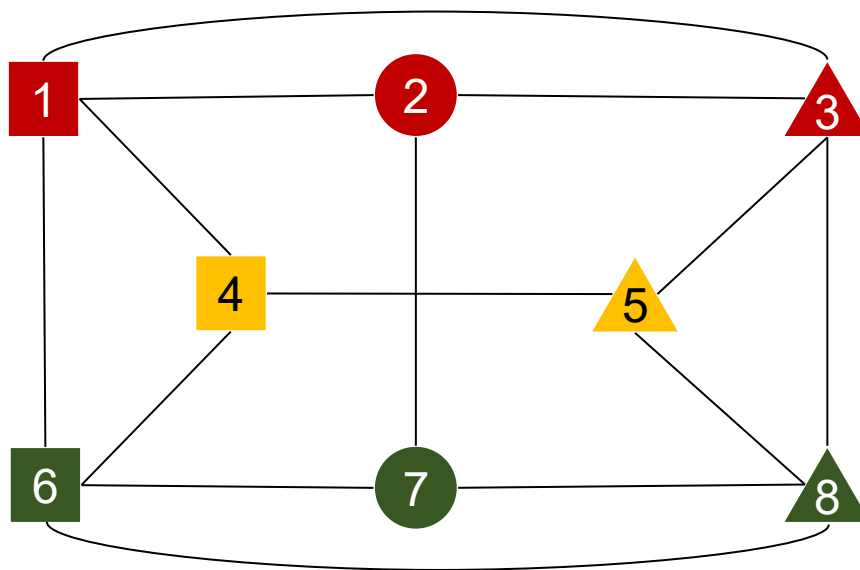
$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:
 - aRa for all $a \in A$ — Reflexivity
 - If aRb , then bRa — Symmetry
 - ~~If aRb and bRc , then aRc~~

Compatible Relations (相容关系)

- An **compatible relation** is a relation that is **reflexive and symmetric**.
- Some examples:
 - x has at least one same letter with y
 - $x \cap y \neq \emptyset$
 - x has same color or same shape with y

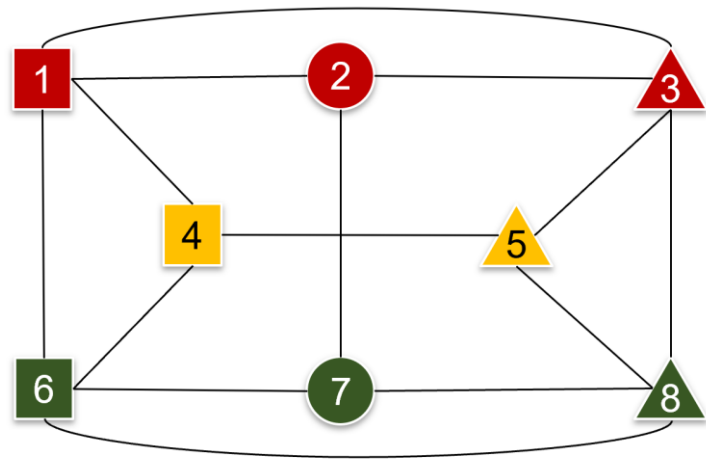
Compatible Relation



xRy if x has same *color* or same *shape* with y

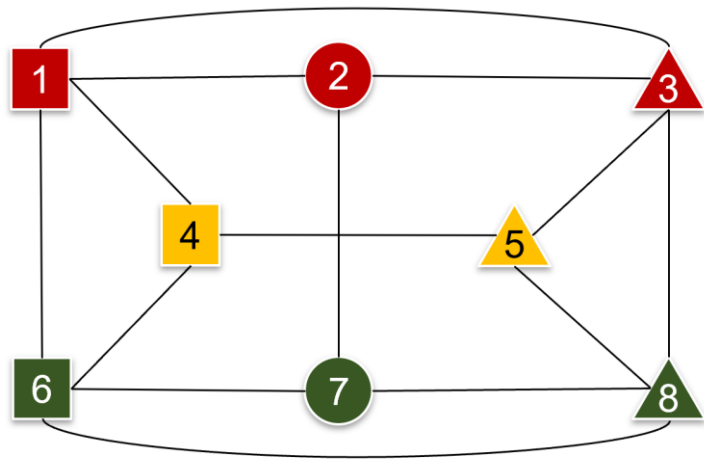
Compatible Class (相容类)

- Given a compatible relation A , C is called **compatible class** if:
 - $C \subseteq A$
 - $(x \in A) \wedge (\forall y)((y \in C) \rightarrow xRy)$
- Examples:
 - $\{1,4,6\}$
 - $\{1,4\}$
 - $\{3,8\}$
 - ...



Maximal Compatible Class(极大相容类)

- A compatible class C is called **maximal compatible class** if C is not the subset of any compatible class, denote by C_R .
- Examples:
 - $\{1,4,6\}$
 - $\{1,2,3\}$
 - $\{2,7\}$
 - $\{3,5,8\}$
 - ...

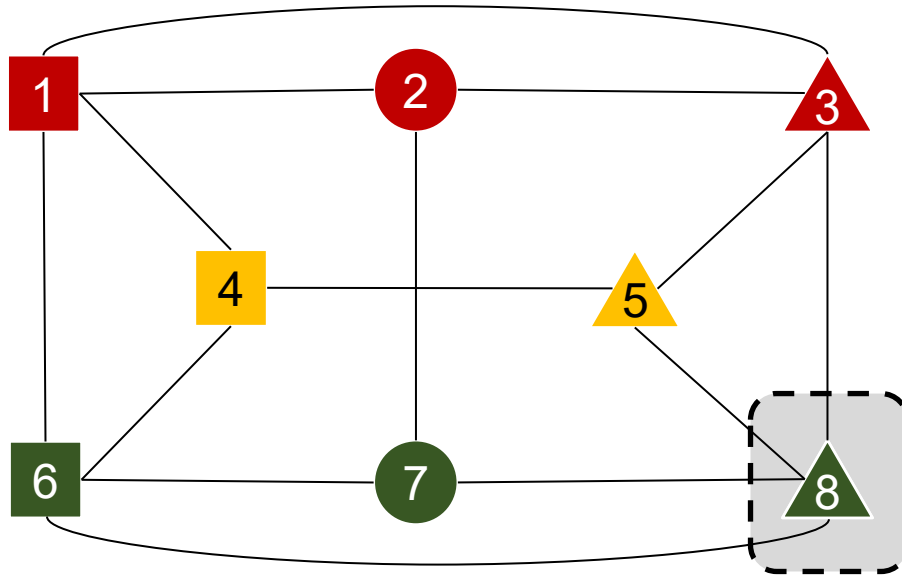


Maximal Compatible Class

For any compatible class C on set A , there exists a maximal compatible class C_R such that $C \subseteq C_R$

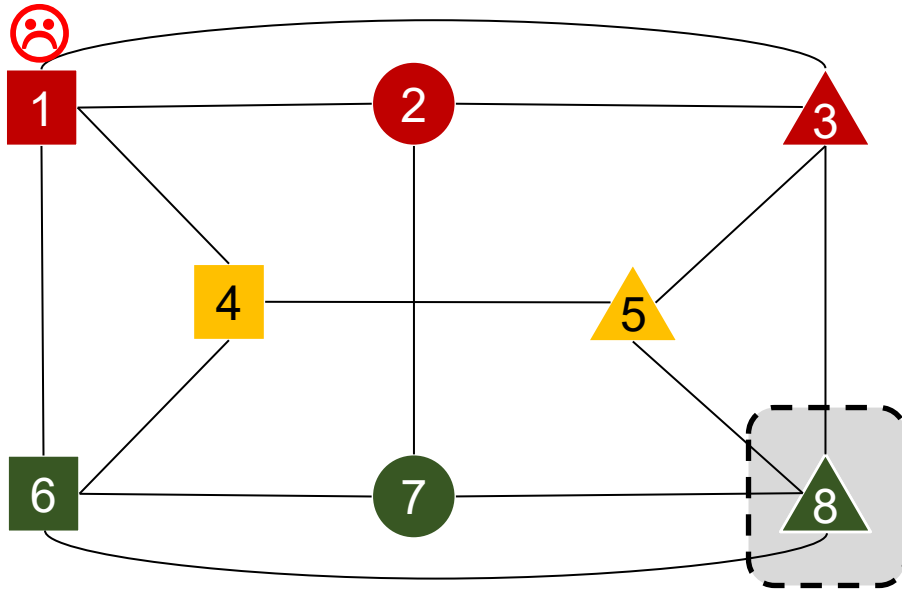
- We can generate a sequence such that $C_0 \subseteq C_1 \subseteq C_2 \dots \subseteq C_R$
 - $C_0 = C$
 - $C_{i+1} = C_i \cup \{a_j\}$, where j is the minimal index such that $a_j \notin C_i \wedge (x \in C_i \rightarrow a_j R x)$

Maximal Compatible Class



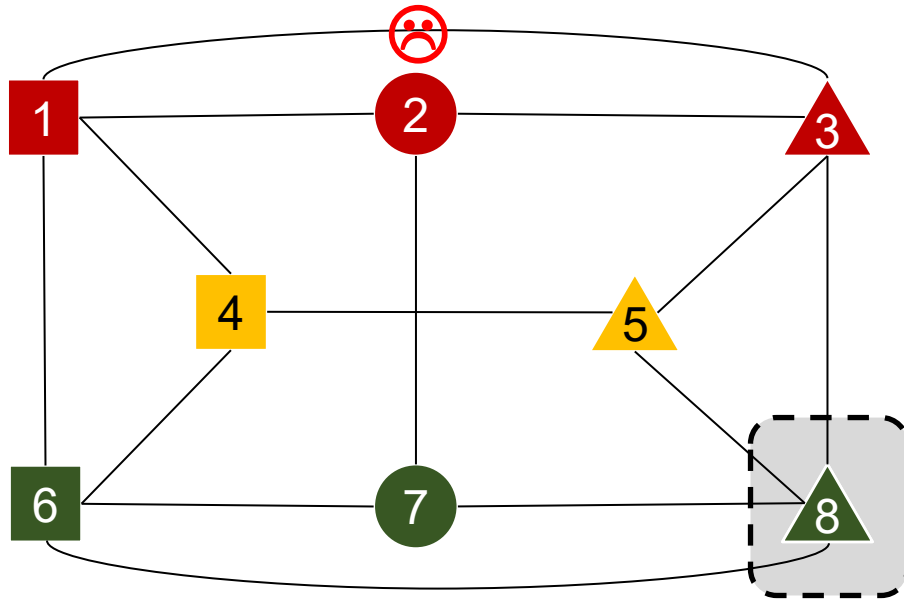
$$C_0 = C = \{8\}$$

Maximal Compatible Class



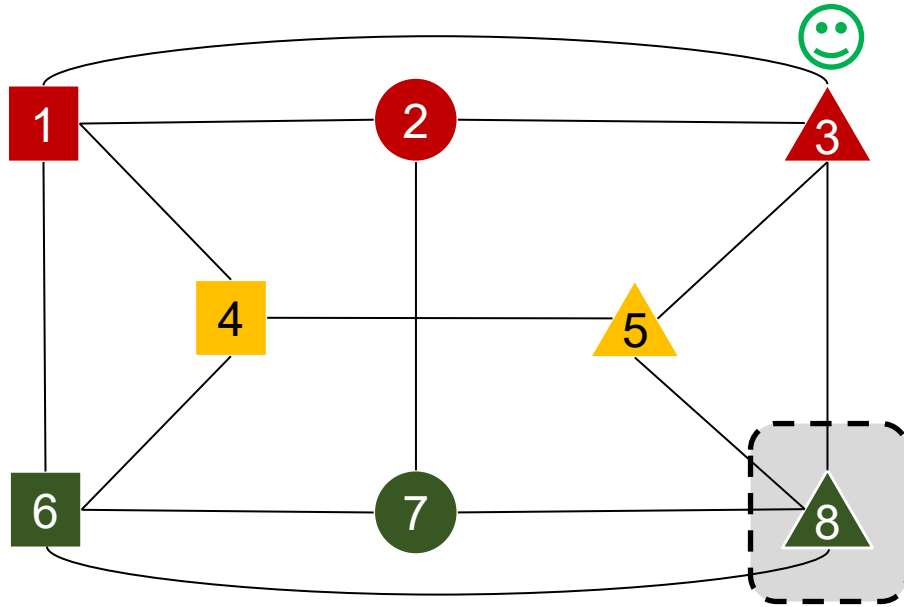
$$C_0 = C = \{8\}$$

Maximal Compatible Class



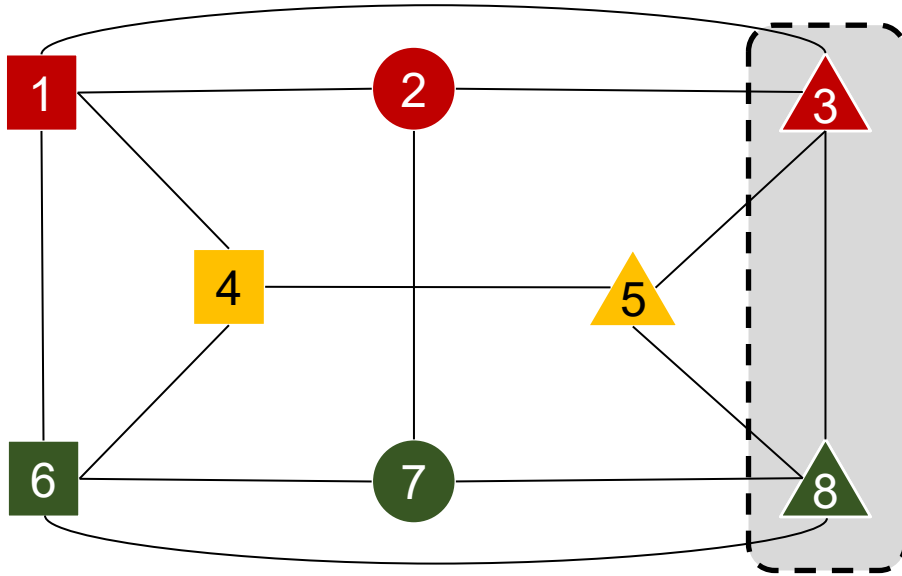
$$C_0 = C = \{8\}$$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

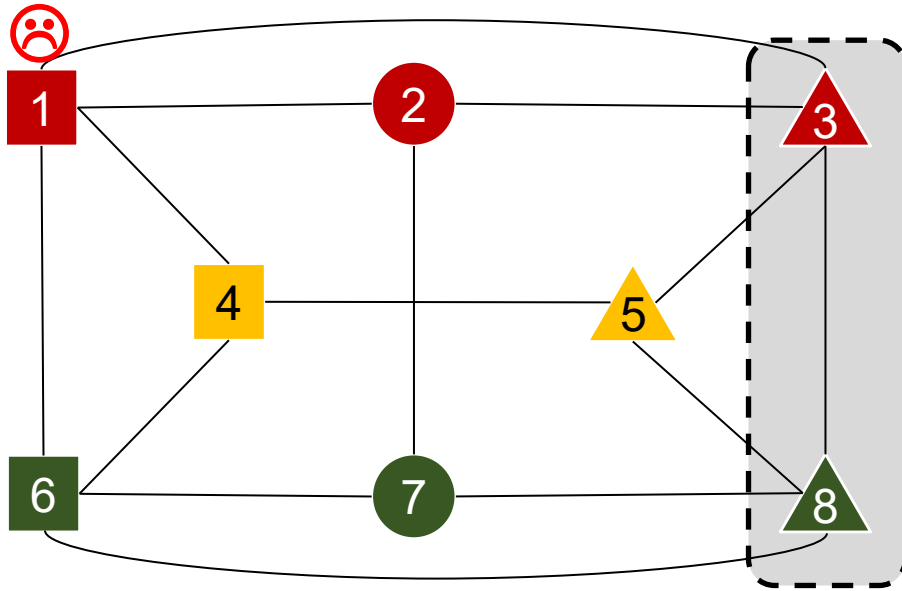
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

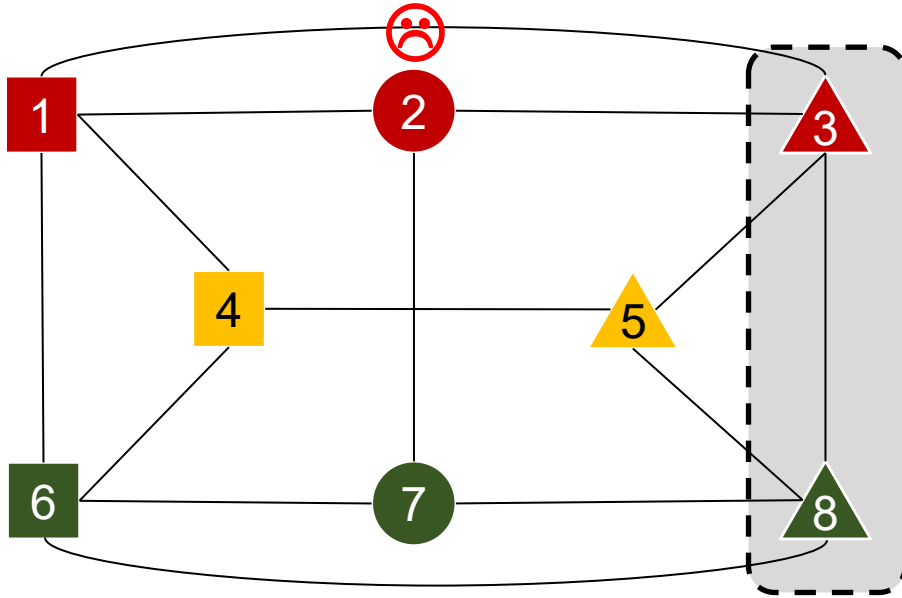
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

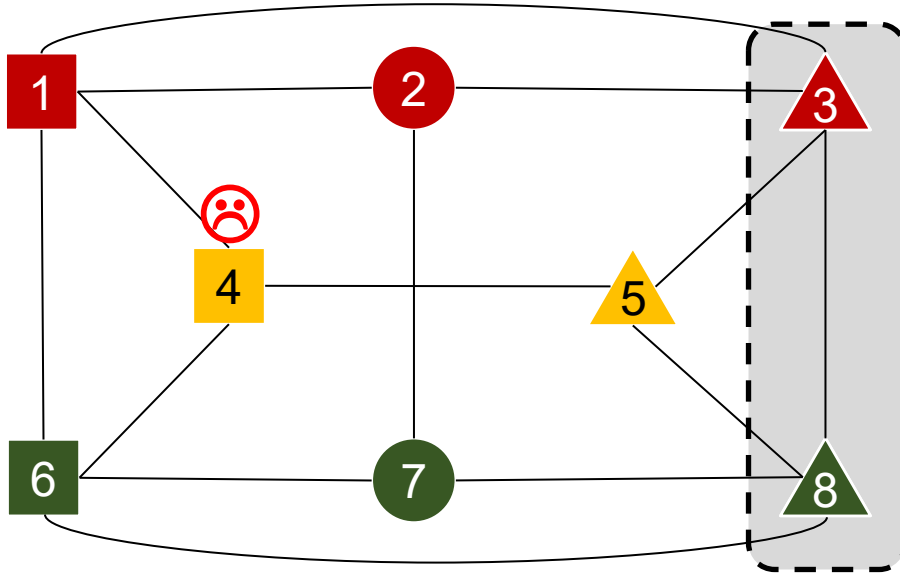
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

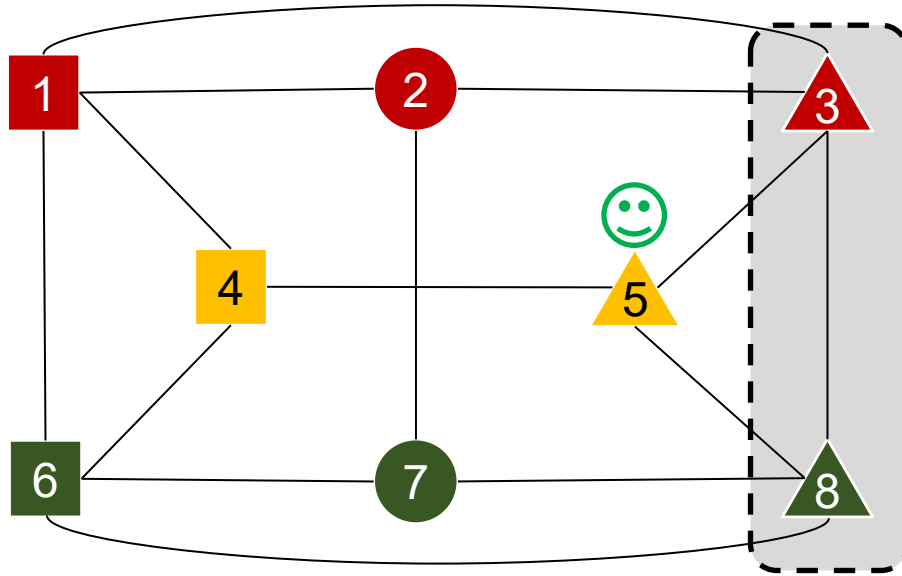
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

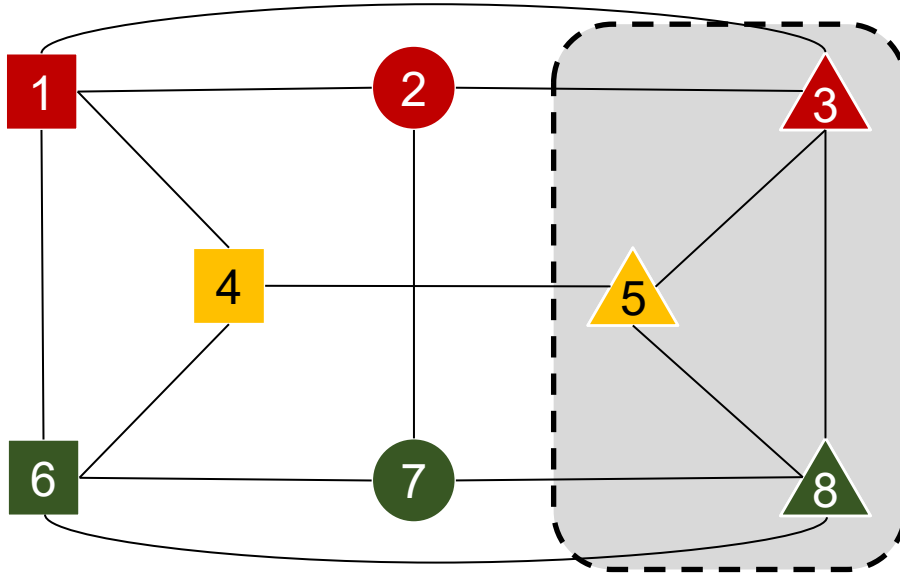
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

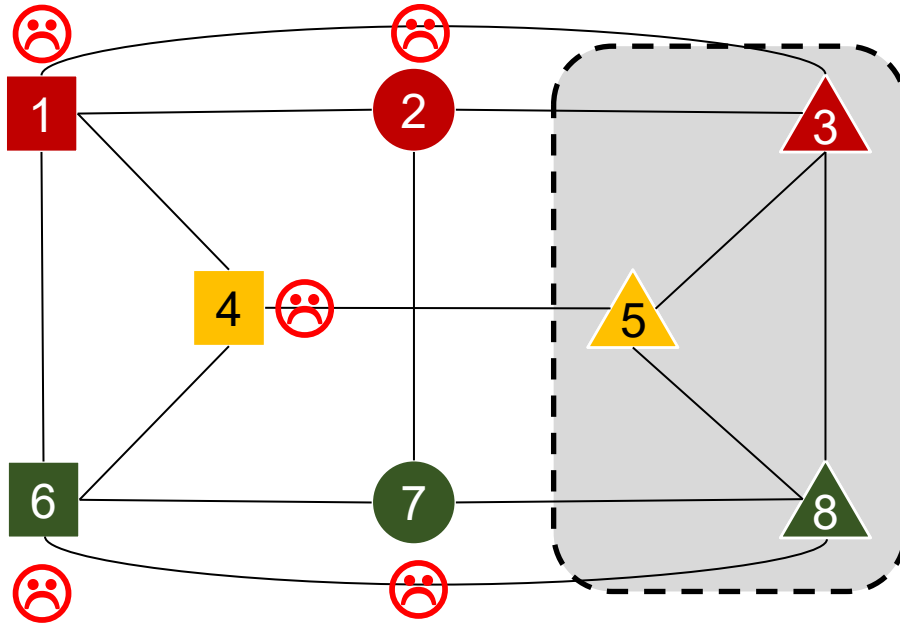
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$

$$C_R = C_2 = \{3, 5, 8\}$$

Compatible Relation and Covers

- Given a compatible relation R on a set A , the set of all maximal compatible classes is a cover of A , called the complete cover (完全覆盖) of A , denote by $C_R(A)$.
 - $C_R(A)$ is unique.
- A compatible relation of A can generate a complete cover of A
- A cover of A can generate a compatible relation of A
- Two different covers can generate the same compatible relation.



Next

Capture the structure of a problem ⟨continued⟩

- Partial Order, total order.
- Closures

Other fundamental tools.

- Functions